Solution to a radial diffusion model with polynomial decaying flux at the boundary

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Introduction

In order to plan a successful secondary recovery project sufficient reliable information concerning the nature of the reservoir-fluid system must be available.

**Heterogeneous medium:**

\[
\begin{align*}
\omega \frac{\partial^2 p_{2D}}{\partial t_D} &= \frac{\partial^2 p_{2D}}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_2}{\partial r_D} - (1 - \omega) \frac{\partial p_{1D}}{\partial t_D} & (1) \\
(1 - \omega) \frac{\partial p_{1D}}{\partial t_D} &= \lambda (p_{2D} - p_{1D}) & (2)
\end{align*}
\]
Introduction

Diffusion in fractal medium:

\[
c \frac{\partial p}{\partial t} = k \frac{1}{\mu r^{D_f-1}} \frac{\partial}{\partial r} \left( r^{D_f-1-\theta} \frac{\partial p}{\partial r} \right)
\]  \hspace{1cm} (3)

where \( k = k_0 r^{-\theta} \).

Homogeneous medium:

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t},
\]  \hspace{1cm} (4)
Chicontepec shows low production rates

- Chicontepec is characterized not so much by a fractal network that extends along particular formations, but rather by an extreme compartmentalization of clusters of the network.
- In Chicontepec, each cluster is embedded in a particular body of sandstone which in itself is surrounded by very low permeability structures.
- Whenever oil is extracted from a sandstone, it therefore depletes soon due to the relative limited volume that the sandstone body can hold.
The basic model

The observed behavior in the flow-rate production or in the pressure can be attributed to many different factors other than fractures such as faults, stratification, and heterogeneities.

Fluid flow:

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t},
\]  

(5)

\[
r_D = \frac{r}{r_w},
\]

\[
t_D = \frac{kt}{\phi \mu c_t r_w^2},
\]

\[
p_D = (p_i - p) \frac{2\pi h k}{q B \mu},
\]
The basic model

With this variables, we have \( r_D = 1 \) and our main equation becomes

\[
\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D}.
\] (6)

We consider either of the following boundary conditions in the well radius

\[
p_D(1, t_D) = 1 \quad \text{production at constant pressure} \quad (7)
\]

\[
\left. \frac{\partial p_D}{\partial r_D} \right|_{r_D=1} = -1 \quad \text{production at constant rate} \quad (8)
\]

Usually this condition is expressed as

\[
\left. \frac{\partial p_D}{\partial r_D} \right|_{r_D=R_D} = 0,
\]
The basic model

Chicontepec:
The oil depletes before the fluxes equilibrate albeit with the flux at the boundary being very small but non-zero. Boundary approximates the waiting times of a sub-diffusive flow outside the sandstone:

\[
\left. \frac{\partial p_D}{\partial r_D} \right|_{r_D=R_D} = -\epsilon(1 - t_D^{-\alpha}),
\]

Finally, let

\[
p_D(r_D, 0) = 0.
\]
Figure: \( p_D \) vs \( t_D \) for a closed boundary reservoir located at \( R_D = 10 \).

\[
p_D(1, t_D) = \frac{2}{R^2 - 1} t_D + \frac{R_D^2}{R^2 - 1} \ln(R_D) - \frac{1}{2} + 2 \sum_{i=1}^{\infty} \frac{e^{-u_i^2 t_D}}{u_i^2} \frac{J_1^2(R_D u_i)}{(J_1^2(R_D u_i) - J_0^2(u_i))}. \tag{11}
\]
A model with variable flux at the boundary

Interior boundary condition

\[
\left( \frac{\partial p_D}{\partial r_D} \right)_{r_D=1} = -1 , \quad t_D > 0 .
\]

and exterior boundary condition

\[
\left( \frac{\partial p_D}{\partial r_D} \right)_{r_D=R_D} = H(t_D - 1)\epsilon(1 - t_D^{-\alpha})
\]

\[
\left( \frac{\partial p_D}{\partial r_D} \right)_{r_D=R_D} = \begin{cases} 
0, & t_D \leq 1 \\
\epsilon(1 - t_D^{-\alpha}), & t_D > 1
\end{cases}
\]

(12)

\[
\bar{p}_D(r_D, s) = \frac{K_1(R_D\sqrt{s}) I_0(r_D\sqrt{s}) + I_1(R_D\sqrt{s}) K_0(r_D\sqrt{s})}{s^3} + \frac{I_1(R_D\sqrt{s}) K_1(\sqrt{s}) - I_1(\sqrt{s}) K_1(R_D\sqrt{s})}{s^2} + \bar{f}(s),
\]

(13)
The production rate decreases very fast and the reservoir gradually fills.

Figure: The influence of $\epsilon$ on the radial model with flux at the boundary can be seen in the curve $p_D$ vs $t_D$ and external radius $R_D = 200$. 
Let \( p_i^k \) the value of the pressure drop at node \((r_i, t_k)\), approximating the equation (2) using a explicit finite difference scheme we have

\[
    p_i^{k+1} = (g - f_i)p_i^{k-1} + (1 - 2g)p_i^k + (f_i + g)p_i^{k+1}
\]

where \( f_i = \frac{\Delta t}{2\Delta r(1 + i\Delta r)} \), \( g = \frac{\Delta t}{\Delta r^2} \) and \( p_i^k \approx p(r_i, t_k) \).

Figure: Numerical solution of the radial diffusion with parameters \( \epsilon = 0.3 \) and \( \alpha = 0.4 \). (a) Plot of \( p_D \) vs \( t_D \) at \( r_D = 1 \). (b) Plot of \( p_D \) vs \( r_D \) at \( t = T \).
Numerical solution

Figure: Numerical solution for different permeabilities for $\alpha=0.4$. 

(a)

(b)

Dimensionless Pressure vs. Dimensionless Time

Dimensionless Pressure vs. Radius
Conservation de la diffusion equation

To validate the numerical solution we calculated the analytical integral over $t_D$ and $r_D$ and verify the conservation of flux in the system. Finally we integrate respect to time, where $T$ is the final time of the production reservoir and the total flow in the system is

$$\int_0^T \frac{d}{dt} \int_{r_D=1}^{r_D=R_D} p(r_D, t_D) r_D dr_D = (1 - R_D \epsilon) T - \alpha R_D \epsilon T^{1-\alpha}.$$ 

**Figure:** Numerical approximation for different mesh sizes where $\epsilon=0.15$ and $\alpha=0.4$. 
Conservation of the diffusion equation

We compare the numerical solution with different mesh size with the following parameters: \( R_D = 200, \ T = 10000, \ \epsilon = 0.15 \) and \( \alpha = 0.4 \), where \( \Delta t = 0.0005 \).

<table>
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<tr>
<th>Mesh size</th>
<th>Numerical integral</th>
<th>Relative error</th>
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<tr>
<td>100</td>
<td>37609.10</td>
<td>3.43</td>
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<td>14450.80</td>
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<tr>
<td>6000</td>
<td>9054.84</td>
<td>0.067</td>
</tr>
</tbody>
</table>

**Table:** Relative errors with different mesh size with \( \Delta t = 0.0005 \).

**Analytical integral:** 8484.0928
Concluding remarks

- We obtain a radial diffusion model for variable flux at the boundary to represent the behavior of Chicontepec.

**Future work:**

- Obtain the analytical solution for different values of $r_D$ and compare with the numerical solution.
- Calculate the semi-log derivative for long times to apply the model to test pressure.
References


